

# Information Geometry, Inference Methods and Chaotic Energy Levels Statistics

Carlo Cafaro\*

*Department of Physics, State University of New York at Albany-SUNY,  
1400 Washington Avenue, Albany, NY 12222, USA*

In this Letter, we propose a novel information-geometric characterization of chaotic (integrable) energy level statistics of a quantum antiferromagnetic Ising spin chain in a tilted (transverse) external magnetic field. Finally, we conjecture our results might find some potential physical applications in quantum energy level statistics.

PACS numbers: 02.50.Tt- Inference methods; 02.40.Ky- Riemannian geometry; 02.50.Cw- Probability theory; 05.45.-a- Nonlinear dynamics and chaos

## I. INTRODUCTION

Research on complexity [1] has created a new set of ideas on how very simple systems may give rise to very complex behaviors. Moreover, in many cases, the "laws of complexity" have been found to hold universally, caring not at all for the details of the system's constituents. Chaotic behavior is a particular case of complex behavior and it will be the object of the present work.

In this Letter we make use of the so-called Entropic Dynamics (ED) [2]. ED is a theoretical framework that arises from the combination of inductive inference (Maximum Entropy Methods (ME), [3]) and Information Geometry (IG) [4]. The most intriguing question being pursued in ED stems from the possibility of deriving dynamics from purely entropic arguments. This is clearly valuable in circumstances where microscopic dynamics may be too far removed from the phenomena of interest, such as in complex biological or ecological systems, or where it may just be unknown or perhaps even nonexistent, as in economics. It has already been shown that entropic arguments do account for a substantial part of the formalism of quantum mechanics, a theory that is presumably fundamental [5]. Perhaps the fundamental theories of physics are not so fundamental; they may just be consistent, objective ways to manipulate information. Following this line of thought, we extend the applicability of ED to temporally-complex (chaotic) dynamical systems on curved statistical manifolds and identify relevant measures of chaoticity of such an information geometrodynamical approach to chaos (IGAC).

The layout of this Letter is as follows. In section II, we present an introduction to the main features of our IGAC. In section III, using our information geometrodynamical approach to chaos and following the results provided by Prosen [6], we propose a novel information-geometric characterization of chaotic (integrable) energy level statistics of a quantum antiferromagnetic Ising chain in a tilted (transverse) external magnetic field. We emphasize that we have omitted technical details that will appear elsewhere. However, some applications of our IGAC to low dimensional chaotic systems can be found in our previous articles [7, 8, 9, 10]. Finally, in section IV we present our final remarks.

## II. THEORETICAL STRUCTURE OF THE IGAC

IGAC arises as a theoretical framework to study chaos in informational geodesic flows describing, physical, biological or chemical systems. It is the information-geometric analogue of conventional geometrodynamical approaches [11, 12] where the classical configuration space  $\Gamma_E$  is being replaced by a statistical manifold  $\mathcal{M}_S$  with the additional possibility of considering chaotic dynamics arising from non conformally flat metrics (the Jacobi metric is always conformally flat, instead). It is an information-geometric extension of the Jacobi geometrodynamics (the geometrization of a Hamiltonian system by transforming it to a geodesic flow [13]). The reformulation of dynamics in terms of a geodesic problem allows the application of a wide range of well-known geometrical techniques in the investigation of the solution space and properties of the equation of motion. The power of the Jacobi reformulation is that all of the dynamical information is collected into a single geometric object in which all the available manifest symmetries are retained- the manifold on which geodesic flow is induced. For example, integrability of the system is connected with existence of Killing vectors and tensors on this manifold. The sensitive dependence of trajectories on initial conditions, which is

---

\*Electronic address: carlocafaro2000@yahoo.it

a key ingredient of chaos, can be investigated from the equation of geodesic deviation. In the Riemannian [11] and Finslerian [12] (a Finsler metric is obtained from a Riemannian metric by relaxing the requirement that the metric be quadratic on each tangent space) geometrodynamical approach to chaos in classical Hamiltonian systems, an active field of research concerns the possibility of finding a rigorous relation among the sectional curvature, the Lyapunov exponents, and the Kolmogorov-Sinai dynamical entropy (i. e. the sum of positive Lyapunov exponents) [14]. Using information-geometric methods, we have investigated in some detail the above still open research problem [8]. One of the goals of this Letter is that of representing an additional step forward in that research direction.

### A. General Formalism of the IGAC

The IGAC is an application of ED to complex systems of arbitrary nature. ED is a form of information-constrained dynamics built on curved statistical manifolds  $\mathcal{M}_S$  where elements of the manifold are probability distributions  $\{P(X|\Theta)\}$  that are in a one-to-one relation with a suitable set of macroscopic statistical variables  $\{\Theta\}$  that provide a convenient parametrization of points on  $\mathcal{M}_S$ . The set  $\{\Theta\}$  is called the *parameter space*  $\mathcal{D}_\Theta$  of the system.

In what follows, we schematically outline the main points underlying the construction of an arbitrary form of entropic dynamics. First, the microstates of the system under investigation must be defined. For the sake of simplicity, we assume the system is characterized by an  $l$ -dimensional microspace with microstates  $\{X\}$ . The main goal of an ED model is that of inferring "macroscopic predictions" in the absence of detailed knowledge of the microscopic nature of arbitrary complex systems. More explicitly, with "macroscopic prediction" we mean the knowledge of the probability function statistical parameters (expectation values, variances, etc.). Once the microstates have been defined, we have to select the relevant information about such microstates. In other words, we have to select the *macrospace* of the system. It is worthwhile mentioning that the coexistence of macroscopic dynamics with microscopic dynamics for a given physical (biological, chemical) system from a dynamical and statistical point of view has always been a very important object of investigation [15]. For example, in certain fluid systems showing Rayleigh-Benard convection [16], the macroscopic chaotic behavior (macroscopic chaos) is a manifestation of the underlying molecular interaction of a very large collection of molecules (microscopic or molecular chaos). Moreover, macroscopic chaos arising from an underlying microscopic chaotic molecular behavior has been observed in chemical reactions. In order to study underlying dynamics in rate equations of chemical reactions, a mesoscopic description has been adopted, which is given by a set of transition probabilities among chemicals. In such a description, the underlying dynamics of macroscopic motion is that one of stochastic processes and the evolution of probability distribution of each chemical is investigated [17].

For the sake of the argument, we assume that the degrees of freedom  $\{x_k\}$  of the microstates  $\{X\}$  are Gaussian-distributed. They are defined by  $2l$ -information constraints, for example their expectation values  $\mu_k$  and variances  $\sigma_k$ .

$$\langle x_k \rangle \equiv \mu_k \text{ and } \left( \langle (x_k - \langle x_k \rangle)^2 \rangle \right)^{\frac{1}{2}} \equiv \sigma_k. \quad (1)$$

In addition to information constraints, each 1-dimensional Gaussian distribution  $p_k(x_k|\mu_k, \sigma_k)$  of each degree of freedom  $x_k$  must satisfy the usual normalization conditions,

$$\int_{-\infty}^{+\infty} dx_k p_k(x_k|\mu_k, \sigma_k) = 1 \quad (2)$$

where

$$p_k(x_k|\mu_k, \sigma_k) = (2\pi\sigma_k^2)^{-\frac{1}{2}} \exp\left(-\frac{(x_k - \mu_k)^2}{2\sigma_k^2}\right). \quad (3)$$

(More correctly, we should label the Gaussian probability distribution in (3) as  $P(x_k|\mu_k, \sigma_k)$ , where  $P$  indicates a specific Gaussian distribution selected from a generic parametric family of distributions). Once the microstates have been defined and the relevant (linear or nonlinear) information constraints selected, we are left with a set of  $l$ -

dimensional vector probability distributions  $p(X|\Theta) = \prod_{k=1}^l p_k(x_k|\mu_k, \sigma_k)$  encoding the relevant available information

about the system where  $X$  is the  $l$ -dimensional microscopic vector with components  $(x_1, \dots, x_l)$  and  $\Theta$  is the  $2l$ -dimensional macroscopic vector with coordinates  $(\mu_1, \dots, \mu_l; \sigma_1, \dots, \sigma_l)$ . The set  $\{\Theta\}$  defines the  $2l$ -dimensional space

of macrostates of the system, the statistical manifold  $\mathcal{M}_S$ . A measure of distinguishability among macrostates is obtained by assigning a probability distribution  $P(X|\Theta) \ni \mathcal{M}_S$  to each macrostate  $\Theta$ . Assignment of a probability distribution to each state endows  $\mathcal{M}_S$  with a metric structure. Specifically, the Fisher-Rao information metric  $g_{\mu\nu}(\Theta)$  [4],

$$g_{\mu\nu}(\Theta) = \int dX p(X|\Theta) \partial_\mu \log p(X|\Theta) \partial_\nu \log p(X|\Theta), \quad (4)$$

with  $\mu, \nu = 1, \dots, 2l$  and  $\partial_\mu = \frac{\partial}{\partial \Theta^\mu}$ , defines a measure of distinguishability among macrostates on  $\mathcal{M}_S$ . The statistical manifold  $\mathcal{M}_S$ ,

$$\mathcal{M}_S = \left\{ p(X|\Theta) = \prod_{k=1}^l p_k(x_k|\mu_k, \sigma_k) \right\}, \quad (5)$$

is defined as the set of probabilities  $\{p(X|\Theta)\}$  described above where  $X \in \mathbb{R}^{3N}$ ,  $\Theta \in \mathcal{D}_\Theta = [\mathcal{I}_\mu \times \mathcal{I}_\sigma]^{3N}$ . The parameter space  $\mathcal{D}_\Theta$  (homeomorphic to  $\mathcal{M}_S$ ) is the direct product of the parameter subspaces  $\mathcal{I}_\mu$  and  $\mathcal{I}_\sigma$ , where (unless specified otherwise)  $\mathcal{I}_\mu = (-\infty, +\infty)_\mu$  and  $\mathcal{I}_\sigma = (0, +\infty)_\sigma$ . It is worthwhile pointing out that the possible chaotic behavior of the set of macrostates  $\{\Theta\}$  is strictly related to the selected relevant information about the set of microstates  $\{X\}$  of the system. In other words, the assumed Gaussian characterization of the degrees of freedom  $\{x_k\}$  of each microstate of the system has deep consequences on the macroscopic behavior of the system itself.

Once  $\mathcal{M}_S$  and  $\mathcal{D}_\Theta$  are defined, the ED formalism provides the tools to explore dynamics driven on  $\mathcal{M}_S$  by entropic arguments. Specifically, given a known initial macrostate  $\Theta^{(\text{initial})}$  (probability distribution), and that the system evolves to a final known macrostate  $\Theta^{(\text{final})}$ , the possible trajectories of the system are examined in the ED approach using ME methods.

We emphasize ED can be derived from a standard principle of least action (of Jacobi type). The geodesic equations for the macrovariables of the Gaussian ED model are given by *nonlinear* second order coupled ordinary differential equations,

$$\frac{d^2 \Theta^\mu}{d\tau^2} + \Gamma_{\nu\rho}^\mu \frac{d\Theta^\nu}{d\tau} \frac{d\Theta^\rho}{d\tau} = 0. \quad (6)$$

The geodesic equations in (6) describe a *reversible* dynamics whose solution is the trajectory between an initial  $\Theta^{(\text{initial})}$  and a final macrostate  $\Theta^{(\text{final})}$ . The trajectory can be equally well traversed in both directions. Given the Fisher-Rao information metric, we can apply standard methods of Riemannian differential geometry to study the information-geometric structure of the manifold  $\mathcal{M}_S$  underlying the entropic dynamics. Connection coefficients  $\Gamma_{\mu\nu}^\rho$ , Ricci tensor  $R_{\mu\nu}$ , Riemannian curvature tensor  $R_{\mu\nu\rho\sigma}$ , sectional curvatures  $\mathcal{K}_{\mathcal{M}_S}$ , scalar curvature  $\mathcal{R}_{\mathcal{M}_S}$ , Weyl anisotropy tensor  $W_{\mu\nu\rho\sigma}$ , Killing fields  $\xi^\mu$  and Jacobi fields  $J^\mu$  can be calculated in the usual way [11, 12].

To characterize the chaotic behavior of complex entropic dynamical systems, we are mainly concerned with the signs of the scalar and sectional curvatures of  $\mathcal{M}_S$ , the asymptotic behavior of Jacobi fields  $J^\mu$  on  $\mathcal{M}_S$ , the existence of Killing vectors  $\xi^\mu$  (or existence of a non-vanishing Weyl anisotropy tensor, the anisotropy of the manifold underlying system dynamics plays a significant role in the mechanism of instability) and the asymptotic behavior of the information-geometro-dynamical entropy (IGE)  $\mathcal{S}_{\mathcal{M}_S}$  (see (9)). It is crucial to observe that true chaos is identified by the occurrence of two features [12]: 1) strong dependence on initial conditions and exponential divergence of the Jacobi vector field intensity, i.e., *stretching* of dynamical trajectories; 2) compactness of the configuration space manifold, i.e., *folding* of dynamical trajectories. The negativity of the Ricci scalar  $\mathcal{R}_{\mathcal{M}_S}$ ,

$$\mathcal{R}_{\mathcal{M}_S} = R_{\mu\nu\rho\sigma} g^{\mu\rho} g^{\nu\sigma} = \sum_{\rho \neq \sigma} \mathcal{K}_{\mathcal{M}_S}(e_\rho, e_\sigma), \quad (7)$$

implies the existence of expanding directions in the configuration space manifold  $\mathcal{M}_S$ . Indeed, since  $\mathcal{R}_{\mathcal{M}_S}$  is the sum of all sectional curvatures of planes spanned by pairs of orthonormal basis elements  $\{e_\rho = \partial_{\Theta^\rho}\}$ , the negativity of the Ricci scalar is only a *sufficient* (not necessary) condition for local instability of geodesic flow. For this reason, the negativity of the scalar provides a *strong* criterion of local instability. Scenarios may arise where negative sectional curvatures are present, but the positive ones could prevail in the sum so that the Ricci scalar is non-negative despite the instability in the flow in those directions. Consequently, the signs of  $\mathcal{K}_{\mathcal{M}_S}$  are of primary significance for the proper characterization of chaos.

A powerful mathematical tool to investigate the stability or instability of a geodesic flow is the Jacobi-Levi-Civita equation (JLC equation) for geodesic spread [11],

$$\frac{D^2 J^\mu}{D\tau^2} + R_{\nu\rho\sigma}^\mu \frac{\partial \Theta^\nu}{\partial \tau} J^\rho \frac{\partial \Theta^\sigma}{\partial \tau} = 0. \quad (8)$$

The JLC-equation covariantly describes how nearby geodesics locally scatter and relates the stability or instability of a geodesic flow with curvature properties of the ambient manifold. Finally, the asymptotic regime of diffusive evolution describing the possible exponential increase of average volume elements on  $\mathcal{M}_s$  provides another useful indicator of dynamical chaoticity. The exponential instability characteristic of chaos forces the system to rapidly explore large areas (volumes) of the statistical manifold. It is interesting to note that this asymptotic behavior appears also in the conventional description of quantum chaos where the entropy (von Neumann) increases linearly at a rate determined by the Lyapunov exponents. The linear increase of entropy as a quantum chaos criterion was introduced by Zurek and Paz [18]. In my information-geometric approach a relevant quantity that can be useful to study the degree of instability characterizing ED models is the information geometrodynamical entropy (IGE) defined as [8],

$$\mathcal{S}_{\mathcal{M}_s}(\tau) \stackrel{\text{def}}{=} \lim_{\tau \rightarrow \infty} \log \mathcal{V}_{\mathcal{M}_s} \text{ with } \mathcal{V}_{\mathcal{M}_s}(\tau) = \frac{1}{\tau} \int_0^\tau d\tau' \left( \int_{\mathcal{M}_s} \sqrt{g} d^{2l} \Theta \right) \quad (9)$$

and  $g = |\det(g_{\mu\nu})|$ . IGE is the asymptotic limit of the natural logarithm of the statistical weight defined on  $\mathcal{M}_s$  and represents a measure of temporal complexity of chaotic dynamical systems whose dynamics is underlined by a curved statistical manifold. In conventional approaches to chaos, the notion of entropy is introduced, in both classical and quantum physics, as the missing information about the systems fine-grained state [19]. For a classical system, suppose that the phase space is partitioned into very fine-grained cells of uniform volume  $\Delta v$ , labelled by an index  $j$ . If one does not know which cell the system occupies, one assigns probabilities  $p_j$  to the various cells; equivalently, in the limit of infinitesimal cells, one can use a phase-space density  $\rho(X_j) = \frac{p_j}{\Delta v}$ . Then, in a classical chaotic evolution, the asymptotic expression of the information needed to characterize a particular coarse-grained trajectory out to time  $\tau$  is given by the Shannon information entropy (measured in bits) [19],

$$\mathcal{S}_{\text{classical}}^{(\text{chaotic})} = - \int dX \rho(X) \log_2(\rho(X) \Delta v) = - \sum_j p_j \log_2 p_j \sim \mathcal{K} \tau. \quad (10)$$

where  $\rho(X)$  is the phase-space density and  $p_j = \frac{v_j}{\Delta v}$  is the probability for the corresponding coarse-grained trajectory.  $\mathcal{S}_{\text{classical}}^{(\text{chaotic})}$  is the missing information about which fine-grained cell the system occupies. The quantity  $\mathcal{K}$  represents the linear rate of information increase and it is called the Kolmogorov-Sinai entropy (or metric entropy) ( $\mathcal{K}$  is the sum of positive Lyapunov exponents,  $\mathcal{K} = \sum_j \lambda_j$ ).  $\mathcal{K}$  quantifies the degree of classical chaos.

### III. APPLICATION OF THE IGAC TO QUANTUM ENERGY LEVEL STATISTICS

The relevant indicators of chaoticity within the IGAC are the Ricci scalar curvature  $\mathcal{R}_{\mathcal{M}_s}$  (or, more correctly, the sectional curvature  $\mathcal{K}_{\mathcal{M}_s}$ ), the Jacobi vector field intensity  $J_{\mathcal{M}_s}$  and the IGE  $\mathcal{S}_{\mathcal{M}_s}$  once the line element on the curved statistical manifold  $\mathcal{M}_s$  underlying the entropic dynamics has been specified. In what follows, selected two special line elements, we focus exclusively on the asymptotic temporal behavior of the IGE  $\mathcal{S}_{\mathcal{M}_s}$  related to the ED arising from them.

We apply the IGAC to study the entropic dynamics on curved statistical manifolds induced by classical probability distributions of common use in the study of regular and chaotic quantum energy level statistics. In doing so, we suggest an information-geometric characterization of a special class of regular and chaotic quantum energy level statistics.

As we said previously, we have omitted technical details that will appear elsewhere. However, our previous works (especially ([8])) may be very useful references in order to clarify the following application. Recall that the theory of quantum chaos (quantum mechanics of systems whose classical dynamics are chaotic) is not primarily related to few-body physics. Indeed, in real physical systems such as many-electron atoms and heavy nuclei, the origin of complex behavior is the very strong interaction among many particles. To deal with such systems, a famous statistical approach has been developed which is based upon the Random Matrix Theory (RMT). The main idea of this approach is to neglect the detailed description of the motion and to treat these systems statistically bearing in mind that the interaction among particles is so complex and strong that generic properties are expected to emerge. Once again, this is exactly the philosophy underlining the ED approach to complex dynamics. It is known that the asymptotic behavior of computational costs and entanglement entropies of integrable and chaotic Ising spin chains are very different [6]. Here Prosen considered the question of time efficiency implementing an up-to-date version of the t-DMRG for a family of Ising spin  $\frac{1}{2}$  chains in arbitrary oriented magnetic field, which undergoes a transition from integrable (transverse Ising) to nonintegrable chaotic regime as the magnetic field is varied. An integrable (regular) Ising chain in a general

homogeneous transverse magnetic field is defined through the Hamiltonian  $\mathcal{H}_{\text{regular}}(0, 2)$ , where

$$\mathcal{H}(h_x, h_y) = \sum_{j=0}^{n-2} \sigma_j^x \sigma_{j+1}^x + \sum_{j=0}^{n-1} (h^x \sigma_j^x + h^y \sigma_j^y). \quad (11)$$

In this case, the computational cost shows a polynomial growth in time,  $D_\varepsilon^{(\text{regular})}(t) \stackrel{\tau \rightarrow \infty}{\propto} \tau$ , while the entanglement entropy is characterized by logarithmic growth,

$$\mathcal{S}_{\text{regular}}(0, 2) = \mathcal{S}_{\text{von Neumann}}^{(0, 2)} \stackrel{\tau \rightarrow \infty}{\propto} c \log \tau + c' \quad (12)$$

The constant  $c$  depends exclusively on the value of the fixed transverse magnetic field intensity  $B_\perp$ , while  $c'$  depends on  $B_\perp$  and on the choice of the initial local operators of finite index used to calculate the operator space entanglement entropy. Instead, a quantum chaotic Ising chain in a general homogeneous tilted magnetic field is defined through the Hamiltonian  $\mathcal{H}_{\text{chaotic}}(1, 1)$ , where  $\mathcal{H}$  is defined in (11). In this case, the computational cost shows an exponential growth in time,  $D_\varepsilon^{(\text{chaotic})}(t) \stackrel{\tau \rightarrow \infty}{\propto} \exp(\mathcal{K}_q \tau)$ , while the entanglement entropy is characterized by linear growth,

$$\mathcal{S}_{\text{chaotic}}(1, 1) = \mathcal{S}_{\text{von Neumann}}^{(1, 1)} \stackrel{\tau \rightarrow \infty}{\propto} \mathcal{K}_q \tau. \quad (13)$$

The quantity  $\mathcal{K}_q$  is a constant, asymptotically independent of the number of indexes of the initial local operators used to calculate the operator space entropy, that depends only on the Hamiltonian evolution and not on the details of the initial state observable or error measures, and can be interpreted as a kind of quantum dynamical entropy.

It is well known the quantum description of chaos is characterized by a radical change in the statistics of quantum energy levels [21]. The transition to chaos in the classical case is associated with a drastic change in the statistics of the nearest-neighbor spacings of quantum energy levels. In the regular regime the distribution agrees with the Poisson statistics while in the chaotic regime the Wigner-Dyson distribution works very well. Uncorrelated energy levels are characteristic of quantum systems corresponding to a classically regular motion while a level repulsion (a suppression of small energy level spacing) is typical for systems which are classically chaotic. A standard quantum example is provided by the study of energy level statistics of an Hydrogen atom in a strong magnetic field. It is known that level spacing distribution (LSD) is a standard indicator of quantum chaos [22]. It displays characteristic level repulsion for strongly nonintegrable quantum systems, whereas for integrable systems there is no repulsion due to existence of conservation laws and quantum numbers. In [6], the authors calculate the LSD of the spectra of  $\mathcal{H}_{\text{regular}}(0, 2)$  and  $\mathcal{H}_{\text{chaotic}}(1, 1)$ . They find that for  $\mathcal{H}_{\text{regular}}(0, 2)$ , the nearest neighbor LSD is described by a Poisson distribution. For  $\mathcal{H}_{\text{chaotic}}(1, 1)$ , they find the nearest neighbor LSD is described by a Wigner-Dyson distribution. Therefore, they conclude that  $\mathcal{H}_{\text{regular}}(0, 2)$  and  $\mathcal{H}_{\text{chaotic}}(1, 1)$  indeed represent generic regular and quantum chaotic systems, respectively. We will encode the relevant information about the spin-chain in a suitable composite-probability distribution taking account of the quantum spin chain and the configuration of the external magnetic field in which they are immersed.

In the ME method [3], the selection of relevant variables is made on the basis of intuition guided by experiment; it is essentially a matter of trial and error. The variables should include those that can be controlled or experimentally observed, but there are cases where others must also be considered. Our objective here is to choose the relevant microvariables of the system and select the relevant information concerning each one of them. In the integrable case, the Hamiltonian  $\mathcal{H}_{\text{regular}}(0, 2)$  describes an antiferromagnetic Ising chain immersed in a transverse homogeneous magnetic field  $\vec{B}_{\text{transverse}} = B_\perp \hat{B}_\perp$  with the level spacing distribution of its spectrum given by the Poisson distribution

$$p_{\text{Poisson}}(x_A | \mu_A) = \frac{1}{\mu_A} \exp\left(-\frac{x_A}{\mu_A}\right), \quad (14)$$

where the microvariable  $x_A$  is the spacing of the energy levels and the macrovariable  $\mu_A$  is the average spacing. The chain is immersed in the *transverse* magnetic field which has just one component  $B_\perp$  in the Hamiltonian  $\mathcal{H}_{\text{regular}}(0, 2)$ . We translate this piece of information in our IGAC formalism, coupling the probability (14) to an exponential bath  $p_B^{(\text{exponential})}(x_B | \mu_B)$  given by

$$p_B^{(\text{exponential})}(x_B | \mu_B) = \frac{1}{\mu_B} \exp\left(-\frac{x_B}{\mu_B}\right), \quad (15)$$

where the microvariable  $x_B$  is the intensity of the magnetic field and the macrovariable  $\mu_B$  is the average intensity. More correctly,  $x_B$  should be the energy arising from the interaction of the *transverse* magnetic field with the spin  $\frac{1}{2}$  particle magnetic moment,  $x_B = |-\vec{\mu} \cdot \vec{B}| = |-\mu B \cos \varphi|$  where  $\varphi$  is the tilt angle. For the sake of simplicity, let us

set  $\mu = 1$ , then in the transverse case  $\varphi = 0$  and therefore  $x_B = B \equiv B_\perp$ . This is our best guess and we justify it by noticing that the magnetic field intensity is indeed a relevant quantity in this experiment (see equation (12)) and its components (intensity) are quantities that are varied during the transitions from integrable to chaotic regimes. In the regular regime, we say the magnetic field intensity is set to a well-defined value  $\langle x_B \rangle = \mu_B$ . Furthermore, notice that the Exponential distribution is identified by information theory as the maximum entropy distribution if only one piece of information (the expectation value) is known. Finally, the chosen composite probability distribution  $P^{(\text{integrable})}(x_A, x_B | \mu_A, \mu_B)$  encoding relevant information about the system is given by,

$$P^{(\text{integrable})}(x_A, x_B | \mu_A, \mu_B) = \frac{1}{\mu_A \mu_B} \exp \left[ - \left( \frac{x_A}{\mu_A} + \frac{x_B}{\mu_B} \right) \right]. \quad (16)$$

Again, we point out that our probability (16) is our best guess and, of course, must be consistent with numerical simulations and experimental data in order to have some merit. We point out that equation (16) is not fully justified from a theoretical point of view, a situation that occurs due to the lack of a systematic way to select the relevant microvariables of the system (and to choose the appropriate information about such microvariables). Let us denote  $\mathcal{M}_S^{\text{integrable}}$  the two-dimensional curved statistical manifold underlying our information geometrodynamics. The line element  $ds_{\text{integrable}}^2$  on  $\mathcal{M}_S^{\text{integrable}}$  is given by,

$$ds_{\text{integrable}}^2 = ds_{\text{Poisson}}^2 + ds_{\text{Exponential}}^2 = \frac{1}{\mu_A^2} d\mu_A^2 + \frac{1}{\mu_B^2} d\mu_B^2. \quad (17)$$

Applying our IGAC (see ([8])) to the line element in (17), we obtain polynomial growth in  $\mathcal{V}_{\mathcal{M}_S}^{\text{integrable}}$  and logarithmic IGE growth,

$$\mathcal{V}_{\mathcal{M}_S}^{(\text{integrable})}(\tau) \stackrel{\tau \rightarrow \infty}{\propto} \exp(c'_{IG} \tau^{c_{IG}}), \quad \mathcal{S}_{\mathcal{M}_S}^{(\text{integrable})}(\tau) \stackrel{\tau \rightarrow \infty}{\propto} c_{IG} \log \tau + c'_{IG}. \quad (18)$$

The quantity  $c_{IG}$  is a constant proportional to the number of Exponential probability distributions in the composite distribution used to calculate the IGE and  $c'_{IG}$  is a constant that depends on the values assumed by the statistical macrovariables  $\mu_A$  and  $\mu_B$ . Equations in (18) may be interpreted as the information-geometric analogue of the computational complexity  $D_\epsilon^{\text{(regular)}}(\tau)$  and the entanglement entropy  $\mathcal{S}_{\text{regular}}(0, 2)$  defined in standard quantum information theory, respectively. We cannot state they are the same since we are not fully justifying, from a theoretical standpoint, our choice of the composite probability (16).

In the chaotic case, the Hamiltonian  $\mathcal{H}_{\text{chaotic}}(1, 1)$  describes an antiferromagnetic Ising chain immersed in a tilted homogeneous magnetic field  $\vec{B}_{\text{tilted}} = B_\perp \hat{B}_\perp + B_\parallel \hat{B}_\parallel$  with the level spacing distribution of its spectrum given by the Poisson distribution  $p_{\text{Wigner-Dyson}}(x'_A | \mu'_A)$

$$p_{\text{Wigner-Dyson}}(x'_A | \mu'_A) = \frac{\pi x'_A}{2\mu_A'^2} \exp \left( - \frac{\pi x_A'^2}{4\mu_A'^2} \right). \quad (19)$$

where the microvariable  $x'_A$  is the spacing of the energy levels and the macrovariable  $\mu'_A$  is the average spacing. The chain is immersed in the *tilted* magnetic vector field which has two components  $B_\perp$  and  $B_\parallel$  in the Hamiltonian  $\mathcal{H}_{\text{chaotic}}(1, 1)$ . We translate this piece of information in our IGAC formalism, coupling the probability (19) to a Gaussian  $p_B^{(\text{Gaussian})}(x'_B | \mu'_B, \sigma'_B)$  given by,

$$p_B^{(\text{Gaussian})}(x'_B | \mu'_B, \sigma'_B) = \frac{1}{\sqrt{2\pi\sigma_B'^2}} \exp \left( - \frac{(x'_B - \mu'_B)^2}{2\sigma_B'^2} \right). \quad (20)$$

where the microvariable  $x'_B$  is the intensity of the magnetic field, the macrovariable  $\mu'_B$  is the average intensity of the magnetic energy arising from the interaction of the *tilted* magnetic field with the spin  $\frac{1}{2}$  particle magnetic moment, and  $\sigma'_B$  is its covariance: during the transition from the integrable to the chaotic regime, the magnetic field intensity is being varied (experimentally). It is being tilted and its two components ( $B_\perp$  and  $B_\parallel$ ) are being varied as well. Our best guess based on the experimental mechanism that drives the transitions between the two regimes is that magnetic field intensity (actually the microvariable  $\mu B \cos \varphi$ ) is Gaussian-distributed (two macrovariables) during this change. In the chaotic regime, we say the magnetic field intensity is set to a well-defined value  $\langle x'_B \rangle = \mu'_B$  with covariance  $\sigma'_B = \sqrt{\langle (x'_B - \langle x'_B \rangle)^2 \rangle}$ . Furthermore, the Gaussian distribution is identified by information theory as the maximum entropy distribution if only the expectation value and the variance are known. Therefore, the chosen

composite probability distribution  $P^{(\text{chaotic})}(x'_A, x'_B | \mu'_A, \mu'_B, \sigma'_B)$  encoding relevant information about the system is given by,

$$P^{(\text{chaotic})}(x'_A, x'_B | \mu'_A, \mu'_B, \sigma'_B) = \frac{\pi (2\pi\sigma_B'^2)^{-\frac{1}{2}}}{2\mu_A'^2} x'_A \exp \left[ - \left( \frac{\pi x_A'^2}{4\mu_A'^2} + \frac{(x'_B - \mu'_B)^2}{2\sigma_B'^2} \right) \right]. \quad (21)$$

Let us denote  $\mathcal{M}_S^{(\text{chaotic})}$  the three-dimensional curved statistical manifold underlying our information geometrodynamics. The line element  $ds_{\text{chaotic}}^2$  on  $\mathcal{M}_S^{(\text{chaotic})}$  is given by,

$$ds_{\text{chaotic}}^2 = ds_{\text{Wigner-Dyson}}^2 + ds_{\text{Gaussian}}^2 = \frac{4}{\mu_A'^2} d\mu_A'^2 + \frac{1}{\sigma_B'^2} d\mu_B'^2 + \frac{2}{\sigma_B'^2} d\sigma_B'^2. \quad (22)$$

Applying our IGAC (see ([8])) to the line element in (22), we obtain exponential growth for  $\mathcal{V}_{\mathcal{M}_s}^{\text{chaotic}}$  and linear IGE growth

$$\mathcal{V}_{\mathcal{M}_s}^{(\text{chaotic})}(\tau) \stackrel{\tau \rightarrow \infty}{\propto} C_{IG} \exp(\mathcal{K}_{IG}\tau), \quad \mathcal{S}_{\mathcal{M}_s}^{(\text{chaotic})}(\tau) \stackrel{\tau \rightarrow \infty}{\propto} \mathcal{K}_{IG}\tau. \quad (23)$$

The constant  $C_{IG}$  encodes information about the initial conditions of the statistical macrovariables parametrizing elements of  $\mathcal{M}_S^{(\text{chaotic})}$ . The constant  $\mathcal{K}_{IG}$ ,

$$\mathcal{K}_{IG} \stackrel{\tau \rightarrow \infty}{\approx} \frac{d\mathcal{S}_{\mathcal{M}_s}(\tau)}{d\tau} \stackrel{\tau \rightarrow \infty}{\approx} \lim_{\tau \rightarrow \infty} \left[ \frac{1}{\tau} \log \left( \left\| \frac{J_{\mathcal{M}_s}(\tau)}{J_{\mathcal{M}_s}(0)} \right\| \right) \right] \stackrel{\text{def}}{=} \lambda_J, \quad (24)$$

is the model parameter of the chaotic system and depends on the temporal evolution of the statistical macrovariables. It plays the role of the standard Lyapunov exponent of a trajectory and it is, in principle, an experimentally observable quantity. The quantity  $J_{\mathcal{M}_s}(\tau)$  is the Jacobi field intensity and  $\lambda_J$  may be considered the information-geometric analogue of the leading Lyapunov exponent in conventional Hamiltonian systems. Given an explicit expression of  $\mathcal{K}_{IG}$  in terms of the observables  $\mu'_A$  and  $\mu'_B$  and  $\sigma'_B$ , a clear understanding of the relation between the IGE (or  $\mathcal{K}_{IG}$ ) and the entanglement entropy (or  $\mathcal{K}_q$ ) becomes the key point that deserves further study. Equations in (23) are the information-geometric analogue of the computational complexity  $D_{\varepsilon}^{(\text{chaotic})}(\tau)$  and the entanglement entropy  $\mathcal{S}_{\text{chaotic}}(1, 1)$  defined in standard quantum information theory, respectively. This result is remarkable, but deserves a deeper analysis in order to be fully understood.

One of the major limitations of our findings is the lack of a detailed account of the comparison of the theory with experiment. This point will be one of our primary concerns in future works. However, some considerations may be carried out at the present stage. The experimental observables in our theoretical models are the statistical macrovariables characterizing the composite probability distributions. In the integrable case, where the coupling between a Poisson distribution and an Exponential one is considered,  $\mu_A$  and  $\mu_B$  are the experimental observables. In the chaotic case, where the coupling between a Wigner-Dyson distribution and a Gaussian is considered,  $\mu'_A$  and  $\mu'_B$  and  $\sigma'_B$  play the role of the experimental observables. We believe one way to test our theory may be that of determining a numerical estimate of the leading Lyapunov exponent  $\lambda_{\text{max}}$  or the Lyapunov spectrum for the Hamiltonian systems under investigation directly from experimental data (measurement of a time series) and compare it to our theoretical estimate for  $\lambda_J$  [23]. However, we are aware that it may be extremely hard to evaluate Lyapunov exponents numerically. Otherwise, knowing that the mean values of the positive Lyapunov exponents are related to the Kolmogorov-Sinai (KS) dynamical entropy, we suggest to measure the KS entropy  $\mathcal{K}$  directly from a time signal associated to a suitable combination of our experimental observables and compare it to our indirect theoretical estimate for  $\mathcal{K}_{IG}$  from the asymptotic behaviors of our statistical macrovariables [24]. We are aware that the ground of our discussion is quite qualitative. However, we hope that with additional study, especially in clarifying the relation between the IGE and the entanglement entropy, our theoretical characterization presented in this Letter will find experimental support in the future. Therefore, the statement that our findings may be relevant to experiments verifying the existence of chaoticity and related dynamical properties on a macroscopic level in energy level statistics in chaotic and regular quantum spin chains is purely a conjecture at this stage.

#### IV. CONCLUSIONS

In this Letter, we proposed a theoretical information-geometric framework suitable to characterize chaotic dynamical behavior of complex systems on curved statistical manifolds. Specifically, an information-geometric characterization

of regular and chaotic quantum energy level statistics appearing in a quantum Ising spin chain in external magnetic field was presented. It is worthwhile emphasizing the following points: the statements that spectral correlations of classically integrable systems are well described by Poisson statistics and that quantum spectra of classically chaotic systems are universally correlated according to Wigner-Dyson statistics are conjectures, known as the BGS (Bohigas-Giannoni-Schmit, [25] and BTG (Berry-Tabor-Gutzwiller, [26]) conjectures, respectively. These two conjectures are very important in the study of quantum chaos, however their validity finds some exceptions. Several other cases may be considered. For instance, chaotic systems having a spectrum that does not obey a Wigner-Dyson distribution may be considered. A chaotic system can also have a spectrum following a Poisson, semi-Poisson, or other types of critical statistics [27]. Moreover, integrable systems having a spectrum that does not obey a Poisson distribution may be considered as well. For instance, the Harper model would represent such a situation. Moreover, it is worthwhile pointing out that not every chaotic system characterized by entropy-like quantities growing linearly in time has a spectrum described by a Wigner-Dyson distribution. Well-known examples presenting such a situation are the cat maps [28] and the famous kicked rotator [29] where its spectrum follows a Poisson distribution in cylinder representation and a Wigner-Dyson in torus representation but the properties of entropy-like quantities are the same in both representations (at least classically). All these cases are not discussed in our characterization.

Therefore, at present stage, because of the above considerations and because of the lack of experimental evidence in support of our theoretical construct, we can only conclude that the IGAC might find some potential applications in certain regular and chaotic dynamical systems and this remains only a conjecture. However, we hope that our work convincingly shows that this information-geometric approach may be considered a serious effort trying to provide a unifying criterion of chaos of both classical and quantum varieties, thus deserving further research and developments.

### Acknowledgments

I am grateful to Saleem Ali, Ariel Caticha and Adom Giffin for very useful discussions and for their previous collaborations. I extend thanks to Cedric Beny, Michael Frey and Jeroen Wouters for their interest and/or useful comments on my research during the NIC@QS07 in Erice, Ettore Majorana Centre. Finally, I sincerely thank the two anonymous Referees for constructive criticism and for very helpful suggestions.

- 
- [1] M. Gell-Mann, "What is Complexity", *Complexity*, vol. 1, no. 1 (1995).
  - [2] A. Caticha, "Entropic Dynamics", in *Bayesian Inference and Maximum Entropy Methods in Science and Engineering*, ed. by R.L. Fry, AIP Conf. Proc. **617**, 302 (2002).
  - [3] A. Caticha and R. Preuss, "Maximum entropy and Bayesian data analysis: Entropic prior distributions", *Phys. Rev.* **E70**, 046127 (2004).
  - [4] S. Amari and H. Nagaoka, *Methods of Information Geometry*, American Mathematical Society, Oxford University Press, 2000.
  - [5] A. Caticha, "Insufficient Reason and Entropy in Quantum Theory" *Found. Phys.* **30**, 227 (2000).
  - [6] T. Prosen and M. Znidaric, "Is the efficiency of classical simulations of quantum dynamics related to integrability?", *Phys. Rev.* **E75**, 015202 (2007); T. Prosen and I. Pizorn, "Operator space entanglement entropy in transverse Ising chain", *Phys. Rev.* **A76**, 032316 (2007).
  - [7] C. Cafaro, S. A. Ali and A. Giffin, "An Application of Reversible Entropic Dynamics on Curved Statistical Manifolds", in *Bayesian Inference and Maximum Entropy Methods in Science and Engineering*, ed. by Ali Mohammad-Djafari, AIP Conf. Proc. **872**, 243-251 (2006).
  - [8] C. Cafaro and S. A. Ali, "Jacobi Fields on Statistical Manifolds of Negative Curvature", *Physica* **D234**, 70-80 (2007).
  - [9] C. Cafaro, "Information Geometry and Chaos on Negatively Curved Statistical Manifolds", in *Bayesian Inference and Maximum Entropy Methods in Science and Engineering*, ed. by K. Knuth, et al., AIP Conf. Proc. **954**, 175 (2007).
  - [10] A. Caticha and C. Cafaro, "From Information Geometry to Newtonian Dynamics", in *Bayesian Inference and Maximum Entropy Methods in Science and Engineering*, ed. by K. Knuth, et al., AIP Conf. Proc. **954**, 165 (2007).
  - [11] L. Casetti, C. Clementi, and M. Pettini, "Riemannian theory of Hamiltonian chaos and Lyapunov exponents", *Phys. Rev.* **E54**, 5969-5984 (1996).
  - [12] M. Di Bari and P. Cipriani, "Geometry and Chaos on Riemann and Finsler Manifolds", *Planet. Space Sci.* **46**, 1543 (1998).
  - [13] C. G. J. Jacobi, "Vorlesungen uber Dynamik", Reimer, Berlin (1866).
  - [14] T. Kawabe, "Indicator of chaos based on the Riemannian geometric approach", *Phys. Rev.* **E71**, 017201 (2005); T. Kawabe, "Chaos based on Riemannian geometric approach to Abelian-Higgs dynamical system", *Phys. Rev.* **E67**, 016201 (2003).
  - [15] T. Shibata and K. Kaneto, "Collective Chaos", *Phys. Rev. Lett.* **81**, 4116 (1998).
  - [16] E. N. Lorenz, "Deterministic nonperiodic flow", *J. Atmos. Sci.* **20**, 130 (1963).
  - [17] R. F. Fox and J. Keizer, "Amplification of intrinsic fluctuations by chaotic dynamics in physical systems", *Phys. Rev.* **A43**, 1709 (1991).



- [18] W. H. Zurek and J. P. Paz, "Decoherence, Chaos, and the Second Law", *Phys. Rev. Lett.* **72**, 2508 (1994); "Quantum Chaos: a decoherent definition", *Physica* **D83**, 300 (1995).
- [19] C. M. Caves and R. Schack, "Unpredictability, Information, and Chaos", *Complexity* **3**, 46-57 (1997); A. J. Scott, T. A. Brun, C. M. Caves, and R. Schack, "Hypersensitivity and chaos signatures in the quantum baker's map", *J. Phys.* **A39**, 13405 (2006).
- [20] W. H. Zurek, "Preferred States, Predictability, Classicality and Environment-Induced Decoherence", *Prog. Theor. Phys.* **89**, 281 (1993).
- [21] G. Casati and B. Chirikov, *Quantum Chaos*, Cambridge University Press (1995); M. V. Berry, "Chaotic Behavior in Dynamical Systems", ed. G. Casati (New York, Plenum), 1985; M. Robnik and T. Prosen, "Comment on energy level statistics in the mixed regimes", arXiv: [chao-dyn/9706023](https://arxiv.org/abs/chao-dyn/9706023), (1997).
- [22] F. Haake, "*Quantum Signatures of Chaos*", Springer-Verlag, Berlin (1991) (2nd enlarged edition, 2000).
- [23] A. Wolf et. al. "Determining Lyapunov Exponents from Time Series, *Physica* **D16**, 285-317 (1985); J. Wright, "Method for calculating a Lyapunov exponent", *Phys. Rev.* **A29**, 2924-2927 (1984).
- [24] P. Grassberger and I. Procaccia, "Estimation of the Kolmogorov entropy from a chaotic signal", *Phys. Rev.* **A28**, 2591-2593 (1983).
- [25] O. Bohigas et. al., "Characterization of Chaotic Quantum Spectra and Universality of Level Fluctuation Laws", *Phys. Rev. Lett.* **52**, 1 (1984).
- [26] M. C. Gutzwiller, "*Chaos in Classical and Quantum Mechanics*", Springer-Verlag, New York (1990)
- [27] A. M. Garcia-Garcia and J. Wang, "Universality in quantum chaos and the one parameter scaling theory", arXiv: [0707.3964](https://arxiv.org/abs/0707.3964) (2007); "Anderson Localization in Quantum Chaos: Scaling and Universality", *Acta Physica Polonica* **A112**, 635-653 (2007).
- [28] Y. Gu, "Evidences of classical and quantum chaos in the time evolution of nonequilibrium ensembles", *Phys. Lett.* **A149**, 95-100 (1990); J. P. Keating, "Asymptotic properties of the periodic orbits of the cat maps", *Nonlinearity* **4**, 277-307 (1991); J. P. Keating, "The cat maps: quantum mechanics and classical motion", *Nonlinearity* **4**, 309-341 (1991).
- [29] F. M. Izrailev, "Simple models of quantum chaos: spectrum and eigenfunctions", *Phys. Rep.* **196**, 299-392 (1990).